

Nonexistence of distributional supersolutions of a semilinear elliptic equation with Hardy potential

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Abstract. In this paper we study nonexistence of non-negative distributional supersolutions for a class of semilinear elliptic equations involving inverse-square potentials.

Key Words: Hardy inequality, critical exponent, nonexistence, distributional solutions, Fermi coordinates, Emden-Flower transform.

2000 Mathematics Subject Classification: 35D05, 35J15.

Introduction

Let Ω define a domain of \mathbb{R}^N , $N \geq 3$. In this paper, we study nonnegative functions u satisfying

$$(0.1) \quad -\Delta u - b(x)u \geq u^p \quad \text{in } \mathcal{D}'(\Omega),$$

with $p > 1$, $b \not\equiv 0$ and $b \in L^1_{loc}(\Omega)$ is a singular potential of Hardy-type. More precisely, we are interested in distributional solutions to (0.1), that is, functions $u \in L^p_{loc}(\Omega)$ such that $b(x)u \in L^1_{loc}(\Omega)$ and

$$\int_{\Omega} u(-\Delta \varphi - b(x)\varphi) dx \geq \int_{\Omega} u^p \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega), \varphi \geq 0.$$

The study of nonexistence results of (very) weak solution to problem (0.1) goes back to [4], where the authors were motivated by the failure of the Implicit Function

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Theorem. Further references in this direction are [5], [9], [12], [13]. We also quote [2], [3] [27], [29], [23], [21], [24], [20].

In this paper, we study nonexistence of solutions to (0.1) when $\partial\Omega$ possesses a conical singularity at 0 as well as when $\partial\Omega$ is of class C^2 at 0. Higher dimensional singularity will be also considered.

For any domain Σ in the unit sphere \mathbb{S}^{N-1} we introduce the cone

$$\mathcal{C}_\Sigma := \{r\sigma \in \mathbb{R}^N \mid r > 0, \sigma \in \Sigma\}.$$

We recall that the best constant in the Hardy inequality for functions supported by \mathcal{C}_Σ is given by

$$\mu(\mathcal{C}_\Sigma) := \inf_{u \in C_c^\infty(\mathcal{C}_\Sigma)} \frac{\int_{\mathcal{C}_\Sigma} |\nabla u|^2 dx}{\int_{\mathcal{C}_\Sigma} |x|^{-2} u^2 dx} = \frac{(N-2)^2}{4} + \lambda_1(\Sigma),$$

where $\lambda_1(\Sigma)$ is the first Dirichlet eigenvalue for the Laplace-Beltrami operator on Σ ([17], [28]). For a given radius $R > 0$ we introduce the cone-like domain

$$\mathcal{C}_\Sigma^R := \mathcal{C}_\Sigma \cap B_R = \{r\sigma \mid r \in (0, R), \sigma \in \Sigma\},$$

where B_R is the ball of radius R centered at 0. We study the inequality

$$(0.2) \quad -\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\mathcal{C}_\Sigma^R),$$

with

$$\lambda_1(\Sigma) < c \leq \mu(\mathcal{C}_\Sigma).$$

By homogeneity, an important role is played by

$$\alpha_\Sigma^- := \frac{N-2}{2} - \sqrt{\mu(\mathcal{C}_\Sigma) - c},$$

which is the smallest root of the equation

$$\alpha^2 - (N-2)\alpha + c - \lambda_1(\Sigma) = 0.$$

We notice that the restriction $c \leq \mu(\mathcal{C}_\Sigma)$ is not restrictive (see Remark 1.5 below) and in addition $\alpha_\Sigma^- > 0$ when $c > \lambda_1(\Sigma)$. Finally we define

$$p_\Sigma = 1 + \frac{2}{\alpha_\Sigma^-}.$$

We observe that $p_\Sigma = \frac{N+2}{N-2}$ when $c = \mu(\mathcal{C}_\Sigma)$ while $p_\Sigma > \frac{N+2}{N-2}$ as soon as $c < \mu(\mathcal{C}_\Sigma)$. In [5], the authors have studied the case $\Omega = B_R \setminus \{0\} = \mathcal{C}_{\mathbb{S}^{N-1}}^R$. They proved that (0.2) has a non-trivial solution in $B_R \setminus \{0\}$ if and only if $p < p_{\mathbb{S}^{N-1}}$. Our first result generalizes the nonexistence result in [5] to cone-like domains.

Theorem 0.1 *Let \mathcal{C}_Σ^R be a cone-like domain of \mathbb{R}^N , $N \geq 3$. For $\lambda_1(\Sigma) < c \leq \mu(\mathcal{C}_\Sigma)$, let $u \in L_{loc}^p(\mathcal{C}_\Sigma^R)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\mathcal{C}_\Sigma^R) .$$

If $p \geq p_\Sigma$ then $u \equiv 0$.

Theorem 0.1 improves a part of the nonexistence results obtained in [22], where more regular supersolutions were considered. We notice that the assumption $p \geq p_\Sigma$ is sharp (see the existence result in [22], Theorem 1.2).

We next consider the case where $0 \in \partial\Omega$ with $\partial\Omega$ is smooth at 0 and $b(x) = c|x|^{-2}$.

We define

$$\mu(\Omega) := \inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |x|^{-2} u^2 dx} .$$

Put $\Omega_r := \Omega \cap B_r(0)$. Recently it was proved in [15] that, there exists $r_0 = r_0(\Omega) > 0$ such that for all $r \in (0, r_0)$

$$(0.3) \quad \mu(\Omega_r) = \mu(\mathcal{C}_{\mathbb{S}_+^{N-1}}) = \frac{N^2}{4} ,$$

with \mathbb{S}_+^{N-1} is a hemisphere centered at 0 so that $\mathcal{C}_{\mathbb{S}_+^{N-1}}$ is a half-space. We have obtained:

Theorem 0.2 *Let Ω be a smooth domain of \mathbb{R}^N , $N \geq 3$, with $0 \in \partial\Omega$. Let $r > 0$ small so that (0.3) holds. For $N-1 < c \leq \frac{N^2}{4}$, let $u \in L_{loc}^p(\Omega_r)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega_r) .$$

If $p \geq p_{\mathbb{S}_+^{N-1}}$ then $u \equiv 0$.

Here also the nonexistence of nontrivial solution for $c \in (N - 1, N^2/4]$ is sharp, see Proposition 3.2.

When we consider general domains, we face some obstacles in the restriction of the parameter c . This is due to the fact that $\mu(\Omega)$ is not in general smaller than $N - 1$ for smooth domains Ω , with $0 \in \partial\Omega$, see [17]. A consequence of Theorem 0.2 is:

Corollary 0.3 *Let Ω be a smooth domain of \mathbb{R}^N , $N \geq 3$, with $0 \in \partial\Omega$. Assume that $N - 1 < c \leq \mu(\Omega)$. Suppose that there exists $u \in L_{loc}^p(\Omega)$, $u \geq 0$ such that*

$$-\Delta u - \frac{c}{|x|^2}u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

If $p \geq p_{\mathbb{S}_+^{N-1}}$ then $u \equiv 0$.

In Corollary 0.3 above, we assume that the interval $[N - 1, \mu(\Omega)]$ is not empty. This is not in general true (see Remark 0.6 below). However it holds for various domains or in higher dimensions. Indeed, we first observe that the inequality $\frac{(N-2)^2}{4} < \mu(\Omega) \leq \frac{N^2}{4}$ is valid for every smooth bounded domain Ω with $0 \in \partial\Omega$, see [17]. In particular $\mu(\Omega) > N - 1$ whenever $N \geq 7$. Hence we get:

Corollary 0.4 *Let Ω be a smooth bounded domain of \mathbb{R}^N , $N \geq 7$, with $0 \in \partial\Omega$. Let $N - 1 < c \leq \mu(\Omega)$ and $u \in L_{loc}^p(\Omega)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2}u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

If $p \geq p_{\mathbb{S}_+^{N-1}}$ then $u \equiv 0$.

When Ω is a smooth domain (not necessarily bounded), with $0 \in \partial\Omega$, is contained in the half-space $\mathcal{C}_{\mathbb{S}_+^{N-1}}$ then obviously $\mu(\Omega) = \frac{N^2}{4}$ by (0.3). In particular, thanks to Theorem 0.2, the restriction $N \geq 7$ in Corollary 0.4 and the boundedness of Ω can be removed. Indeed, we have:

Corollary 0.5 *Let Ω be a smooth domain of the half-space $\mathcal{C}_{\mathbb{S}_+^{N-1}}$, $N \geq 3$, with $0 \in \partial\Omega$. Let $N - 1 < c \leq \frac{N^2}{4}$ and $u \in L_{loc}^p(\Omega)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2}u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

If $p \geq p_{\mathbb{S}_+^{N-1}}$ then $u \equiv 0$.

Remark 0.6 According to our argument, the assumption $N - 1 < \mu(\Omega)$ is crucial because it implies that $1 < p_{\mathbb{S}_+^{N-1}} < \infty$ when $c > N - 1$. However it is not valid for every smooth domain. In fact, one can construct a family of smooth bounded domains Ω^ε , for which $\mu(\Omega^\varepsilon) \leq \frac{(N-2)^2}{4} + \varepsilon$, for $\varepsilon > 0$ small, see [17], [16].

Remark 0.7 The conclusion in theorems 0.1, 0.2 still holds when u^p is replaced by $|x|^s u^q$ with $\lambda_1(\Sigma) < c \leq \mu(\mathcal{C}_\Sigma)$. In this case one has to replace p_Σ with $q_\Sigma = 1 + \frac{2-s}{\alpha_\Sigma}$.

We prove our nonexistence results via a linearization argument which were also used in [22]. However when working with weaker notion of solutions, further analysis are required. Our approach is to obtain a quite sharp lower estimate on u in such a way that u^{p-1} is somehow proportional to $b(x)$ and to look the problem as a linear problem: $-\Delta u - b(x)u - u^{p-1}u \geq 0$ in $\mathcal{D}'(\Omega)$. This leads to the inequality (see Lemma 1.4)

$$(0.4) \quad \int_{\Omega} |\nabla \varphi|^2 - \int_{\Omega} b(x) \varphi^2 \geq \int_{\Omega} u^{p-1} \varphi^2 \quad \forall \varphi \in C_c^\infty(\Omega).$$

By using appropriate test functions in (0.4), we were able to contradict the existence of solutions. To lower estimate u , we construct sub-solutions for the operator $L := -\Delta - b(x)$. On the other hand since we are working with "very weak" super-solutions in non-smooth domains, and the operator L does not in general satisfies the maximum principle, we have proved a comparison principle (see Lemma 1.3 in Section 1). We achieve this by requiring L to be coercive. Since in this paper the potential $b(x)$ is of Hardy-type, such coercivity is nothing but improvements of Hardy inequalities. The comparison principle allows us to put below u a more regular function v . Such function v turns out to be a supersolution for L and therefore can be lower estimated by the sub-solutions via standard arguments.

The paper is organized as follows. In Section 1 we prove some preliminary results, which are mainly used in the paper. The proofs of Theorems 0.1, 0.2 will be carried out in Sections 2, 3 respectively. Finally in the last section, we study the problem

$$(0.5) \quad \begin{cases} -\Delta u - \frac{(N-k-2)^2}{4} \frac{1}{\text{dist}(x, \Gamma)^2} q(x) u \geq u^p & \text{in } \mathcal{D}'(\Omega \setminus \Gamma), \\ u \in L_{loc}^p(\Omega \setminus \Gamma), \\ u \not\geq 0, \end{cases}$$

where Γ is a smooth closed submanifold of Ω and q is a nonnegative weight.

1 Preliminaries and comparison lemmata

Let Ω be a bounded open subset of \mathbb{R}^N . In this section we deal with comparison results involving a differential operator of the type

$$-\Delta - b(x),$$

where $b \in L^1_{loc}(\Omega)$ is a given non-negative weight. We shall always assume that $-\Delta - b(x)$ is coercive, in the sense that there exists a constant $C(\Omega) > 0$ such that

$$(1.1) \quad \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} b(x)u^2 dx \geq C(\Omega) \int_{\Omega} u^2 dx \quad \text{for any } u \in C_c^\infty(\Omega).$$

Following [10], we define the space $H(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the scalar product

$$(u, v) \mapsto \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} b(x)uv dx.$$

The scalar product in $H(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle_{H(\Omega)}$.

Clearly $H_0^1(\Omega) \hookrightarrow H(\Omega) \hookrightarrow L^2(\Omega)$ by (1.1), and hence $L^2(\Omega)$ embeds into the dual space $H(\Omega)'$. By the Lax Milligram theorem, for any $f \in L^2(\Omega)$ there exists a unique function $v \in H(\Omega)$ such that

$$-\Delta v - b(x)v = f \quad \text{in } H(\Omega),$$

that is,

$$\langle v, \varphi \rangle_{H(\Omega)} = \int_{\Omega} f \varphi dx \quad \text{for any } \varphi \in H(\Omega).$$

Remark 1.1 *Observe that if $b \in L^\infty(\Omega)$ then $H(\Omega) = H_0^1(\Omega)$ since*

$$C \int_{\Omega} |\nabla u|^2 dx \leq \|u\|_{H(\Omega)}^2 \leq \int_{\Omega} |\nabla u|^2 dx,$$

where the constant $C > 0$ depends only on $C(\Omega)$, and on the L^∞ norm of b .

We start with the following technical result which will be useful in the sequel.

Lemma 1.2 *Let $u \in L^1_{loc}(\Omega)$ be non-negative and $g \in L^2(\Omega)$ such that*

$$-\Delta u \geq g \quad \text{in } \mathcal{D}'(\Omega).$$

Let $v \in H^1_0(\Omega)$ be the solution to

$$-\Delta v = g \quad \text{in } \Omega.$$

Then

$$v \leq u \quad \text{in } \Omega.$$

Proof. For $\varepsilon > 0$, define $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$. Let $\tilde{\Omega}_\varepsilon$ be a smooth open set compactly contained in Ω and containing Ω_ε . Denote by ρ_n the standard mollifier and put $u_n = \rho_n * u$. Then for $\varepsilon > 0$ there exists N_ε such that u_n is smooth in $\tilde{\Omega}_\varepsilon$ up to the boundary for all $n \geq N_\varepsilon$. Consider $v_{\varepsilon,n} \in H^1_0(\tilde{\Omega}_\varepsilon)$ be the solution of $-\Delta v_{\varepsilon,n} = \rho_n * g = g_n$ in $\tilde{\Omega}_\varepsilon$. Clearly $-\Delta(u_n - v_{\varepsilon,n}) \geq 0$ in $\tilde{\Omega}_\varepsilon$ and $u_n - v_{\varepsilon,n} \geq 0$ on $\partial\tilde{\Omega}_\varepsilon$, because u is non-negative. It turns out that $u_n - v_{\varepsilon,n} \geq 0$ in $\tilde{\Omega}_\varepsilon$ by the maximum principle. Letting $v_\varepsilon \in H^1_0(\tilde{\Omega}_\varepsilon)$ be the solution of $-\Delta v_\varepsilon = g$ in $\tilde{\Omega}_\varepsilon$, by Hölder and Poincaré inequalities, we have that $\|v_{\varepsilon,n} - v_\varepsilon\|_{H^1_0(\tilde{\Omega}_\varepsilon)} \leq C\|g_n - g\|_{L^2(\tilde{\Omega}_\varepsilon)}$, with $C > 0$ is a constant independent on n . In particular $v_{\varepsilon,n}$ converges to v_ε in $\tilde{\Omega}_\varepsilon$. Therefore $u \geq v_\varepsilon$ in $\tilde{\Omega}_\varepsilon$. To conclude, it suffices to notice that $v_\varepsilon \rightarrow v$ weakly in $H^1_0(\Omega)$ and pointwise in Ω . \square

We have the following comparison principle.

Lemma 1.3 *Let $u \in L^1_{loc}(\Omega)$ be non-negative with $b(x)u \in L^1_{loc}(\Omega)$ and let $f \in L^2(\Omega)$ with $f \geq 0$ such that*

$$-\Delta u - b(x)u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

Let $v \in H(\Omega)$ be the solution of

$$-\Delta v - b(x)v = f \quad \text{in } H(\Omega).$$

Then

$$v \leq u \quad \text{in } \Omega.$$

Proof. Step 1: We first prove the result if $b \in L^\infty(\Omega)$.

We let $v_0 \in H_0^1(\Omega)$ solving

$$-\Delta v_0 = f \quad \text{in } \Omega.$$

Then $0 \leq v_0 \leq u$ in Ω by Lemma 1.2 and because $f \geq 0$. We define inductively the sequence $v_n \in H_0^1(\Omega)$ by

$$-\Delta v_1 = b(x)v_0 + f \quad \text{in } \Omega, \quad -\Delta v_n = b(x)v_{n-1} + f \quad \text{in } \Omega.$$

Since $b \geq 0$, we have $-\Delta u \geq b(x)v_0 + f$ in $\mathcal{D}'(\Omega)$. Thus using once again Lemma 1.2, we obtain $v_0 \leq v_1 \leq u$ in Ω . By induction, we have

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq u \quad \text{in } \Omega \quad \forall n \in \mathbb{N}.$$

Since $v_{n-1} \leq v_n$ in Ω , we have

$$\int_{\Omega} |\nabla v_n|^2 dx - \int_{\Omega} b(x)|v_n|^2 \leq \int_{\Omega} f(x)v_n dx.$$

By Hölder inequality and (1.1) (see Remark 1.1) v_n is bounded in $H_0^1(\Omega)$. We conclude that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$ which is the unique solution to

$$-\Delta v = b(x)v + f \quad \text{in } \Omega.$$

Since $v_n \rightarrow v$ in $L^2(\Omega)$, we get $v \leq u$ in Ω .

Step 2: Conclusion of the proof.

We put $b_k(x) = \min(b(x), k)$ for every $k \in \mathbb{N}$. We consider $v^k \in H_0^1(\Omega)$ be the unique solution to

$$(1.2) \quad \int_{\Omega} \nabla v^k \nabla \varphi - \int_{\Omega} \min\{b(x), k\} v^k \varphi = \int_{\Omega} f \varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

Thanks to **Step 1**, we have $v^k \leq u$ in Ω .

Next, we check that such a sequence v^k , satisfying (1.2), converges to v in $L^2(\Omega)$ when $k \rightarrow \infty$. Indeed, we have

$$\begin{aligned} \|v^k\|_{H(\Omega)}^2 &\leq \|v^k\|_{H_0^1(\Omega)}^2 - \int_{\Omega} \min\{b(x), k\} |v^k|^2 dx \\ &= \int_{\Omega} f v^k dx \leq C \|v^k\|_{H(\Omega)} \end{aligned}$$

by Hölder inequality and by (1.1), where the constant C depends on f and Ω but not on k . Therefore the sequence v^k is bounded in $H(\Omega)$. We conclude that there exists $\tilde{v} \in H(\Omega)$ such that, for a subsequence, $v^k \rightharpoonup \tilde{v}$ in $H(\Omega)$. Now by (1.2), we have

$$\langle v^k, \varphi \rangle_{H(\Omega)} + \int_{\Omega} (b(x) - \min\{b(x), k\}) v^k \varphi = \int_{\Omega} f \varphi.$$

Since for every $k \geq 1$ and any $\varphi \in C_c^\infty(\Omega)$

$$\left| (b(x) - \min\{b(x), k\}) v^k \varphi \right| \leq (b(x) - \min\{b(x), k\}) u |\varphi| \leq 2b(x)u |\varphi| \in L^1(\Omega),$$

the dominated convergence theorem implies that

$$(1.3) \quad \langle \tilde{v}, \varphi \rangle_{H(\Omega)} = \int_{\Omega} f \varphi \quad \text{for any } \varphi \in C_c^\infty(\Omega).$$

We therefore have that $\tilde{v} = v$ by uniqueness. By (1.3), we have

$$\begin{aligned} \|v - v^k\|_{H(\Omega)}^2 &= \|v^k\|_{H(\Omega)}^2 - \langle v, v^k \rangle_{H(\Omega)} + \langle v, v - v^k \rangle_{H(\Omega)} \\ &= \|v^k\|_{H(\Omega)}^2 - \int_{\Omega} f v^k + \langle v, v - v^k \rangle_{H(\Omega)} \\ &\leq \|v^k\|_{H_0^1(\Omega)}^2 - \int_{\Omega} \min\{b(x), k\} |v^k|^2 dx - \int_{\Omega} f v^k + \langle v, v - v^k \rangle_{H(\Omega)} \\ &= \langle v, v - v^k \rangle_{H(\Omega)}. \end{aligned}$$

We thus obtain

$$C(\Omega) \int_{\Omega} |v - v^k|^2 dx \leq \langle v, v - v^k \rangle_{H(\Omega)} \rightarrow 0$$

by (1.1). Hence $v^k \rightarrow v$ pointwise and thus $v \leq u$ in Ω . □

We conclude this section by pointing out the following Allegretto-Piepenbrink type result which is essentially contained in [18]. A version for distributional solutions is also contained in [[8], Theorem 2.12].

Lemma 1.4 *Let Ω be a domain (possibly unbounded) in \mathbb{R}^N , $N \geq 1$. Let $V \in L_{loc}^1(\Omega)$ and $V > 0$ in Ω . Assume that $u \in L_{loc}^1(\Omega)$, $V(x)u \in L_{loc}^1(\Omega)$ and that u is a non-negative, non-trivial solution to*

$$-\Delta u \geq V(x)u \quad \mathcal{D}'(\Omega).$$

Then

$$\int_{\Omega} |\nabla \phi|^2 dx \geq \int_{\Omega} V(x) \phi^2 dx \quad \text{for any } \phi \in C_c^\infty(\Omega).$$

Proof. Put $V_k(x) = \min\{V(x), k\}$ then Lemma B.1 in [18] yields

$$\int_{\Omega} |\nabla \phi|^2 dx \geq \int_{\Omega} V_k(x) \phi^2 dx \quad \text{for any } \phi \in C_c^\infty(\Omega).$$

To conclude, it suffices to use Fatou's lemma. \square

Remark 1.5 Given Ω any domain in \mathbb{R}^N , $N \geq 1$. Define

$$\mu(\Omega) := \inf_{u \in C_c^\infty(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |x|^{-2} u^2 dx}.$$

Then Lemma 1.4 clearly implies that if $c > \mu(\Omega)$ there is no non-negative and non-trivial $u \in L_{loc}^1(\Omega)$ that satisfies $-\Delta u - \frac{c}{|x|^2} u \geq 0$ in $\mathcal{D}'(\Omega)$.

Suppose that Ω is a smooth bounded domain and that the potential $b(x)$ satisfies

$$\int_{\Omega} |\nabla \varphi|^2 dx - \int_{\Omega} b(x) \varphi^2 dx \geq C(b) \left(\int_{\Omega} |\varphi|^r dx \right)^{\frac{2}{r}} \quad \text{for any } \varphi \in C_c^\infty(\Omega)$$

for some $C(b) > 0$ and $2 < r$. By [[10] Lemma 7.2], we can let $G \in L^1(\Omega \times \Omega)$ be the Green function associated to $-\Delta - b(x)$:

$$\begin{cases} -\Delta G(\cdot, y) - b(x)G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_y denotes the Dirac measure at some $y \in \Omega$. Define

$$\zeta_0(x) := \int_{\Omega} G(x, y) dy$$

which is the $H(\Omega)$ -solution to $-\Delta \zeta_0 - b(x) \zeta_0 = 1$. By using Lemma 1.3 and Lemma 1.4, we can prove the following

Proposition 1.6 Suppose that $\int_{\Omega} \zeta_0^{p+1} dx = \infty$ for some $p > r$ then there is no nonnegative and nontrivial u satisfying $-\Delta u - b(x) u \geq u^p$ in $\mathcal{D}'(\Omega)$.

Proof. If such u exists, it is positive by the maximum principle therefore, we can define $v \in H(\Omega)$ be the solution of $-\Delta v - b(x)v = \min(u^p, 1)$ so that by Lemma 1.3 we have $u \geq v$ in Ω . Thanks to [[10] Corollary 2.4], we have $u \geq v \geq C\zeta_0$. By applying Lemma 1.4 with $V(x) = b(x) + (C\zeta_0)^{p-1}$ we conclude that

$$\infty > \|\zeta_0\|_{H(\Omega)} \geq C^{p+1} \int_{\Omega} \zeta_0^{p+1} dx.$$

□

2 Proof of Theorem 0.1

We state the following lemma which is a consequence of Lemma 1.3 and [[22], Theorem 4.2].

Lemma 2.1 *Let $u \in L^1_{loc}(\Omega)$ be positive and let $f \in L^1_{loc}(\mathcal{C}^r_{\Sigma})$ with $f \not\equiv 0$ such that*

$$-\Delta u - V\left(\frac{x}{|x|}\right) |x|^{-2} u \geq f \quad \text{in } \mathcal{D}'(\mathcal{C}^r_{\Sigma}),$$

where $\|V\|_{L^\infty(\Sigma)} \leq \mu(\mathcal{C}_{\Sigma})$ and $V \geq 0$. Then for every $\tilde{\Sigma} \subset\subset \Sigma$ there exists a constant $C > 0$ such that

$$(2.1) \quad u(x) \geq C|x|^{\frac{2-N}{2} + \sqrt{\frac{(2-N)^2}{4} + \lambda_{1,V}}} \quad \text{in } \mathcal{C}^{r/2}_{\tilde{\Sigma}},$$

where $\lambda_{1,V}$ is the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}\Phi - V\Phi = \lambda_{1,V}\Phi$ on Σ .

Proof. Up to a scaling, we can assume that $r = 1$. We recall the following improved Hardy inequality

$$(2.2) \quad \int_{\mathcal{C}^1_{\Sigma}} |\nabla \varphi|^2 dx - \mu(\mathcal{C}_{\Sigma}) \int_{\mathcal{C}^1_{\Sigma}} |x|^{-2} |\varphi|^2 \geq C_0 \int_{\mathcal{C}^1_{\Sigma}} |\varphi|^2 dx \quad \forall \varphi \in C_c^\infty(\mathcal{C}^1_{\Sigma}),$$

for some $C_0 > 0$ (see for instance [17]). We can therefore pick $v \in H(\mathcal{C}^1_{\Sigma})$ solves

$$(2.3) \quad -\Delta v - V\left(\frac{x}{|x|}\right) |x|^{-2} v = \min(f, 1) \quad \text{in } H(\mathcal{C}^1_{\Sigma}).$$

Then by the maximum principle and Lemma 1.3, we have $0 < v \leq u$ in \mathcal{C}_Σ^1 .

Approximating v by smooth functions compactly supported in \mathcal{C}_Σ^1 with respect to the $H(\mathcal{C}_\Sigma^1)$ -norm, we infer that

$$-\Delta v - V\left(\frac{x}{|x|}\right) |x|^{-2}v = \min(f, 1) \quad \text{in } \mathcal{D}'(\mathcal{C}_\Sigma^1).$$

Elliptic regularity theory then implies that $v \in C_{loc}^{1,\gamma}(\mathcal{C}_\Sigma^1) \subset H_{loc}^1(\mathcal{C}_\Sigma^1)$. By applying [[22], Theorem 4.2] (up to Kelvin transform), we get the lower estimate (2.1) for v and hence for u . □

Proof of Theorem 0.1

Up to a scaling, we can assume that $R = 1$. We argue by contradiction. If $u \neq 0$ then by the maximum principle $u > 0$ in \mathcal{C}_Σ^1 . We will show that appropriate lower bound of u and an application of Lemma 1.4 will lead to a contradiction.

Case 1: $c < \mu(\mathcal{C}_\Sigma)$.

By Lemma 2.1

$$u(x) \geq C_0 |x|^{\frac{2-N}{2} + \sqrt{\mu(\mathcal{C}_\Sigma) - c}} \quad \forall x \in \mathcal{C}_\Sigma^{1/2},$$

where C_0 is a positive constant and $\tilde{\Sigma} \subset \subset \Sigma$. By assumption $u^{p-1}(x)|x|^2 \geq C_0^{p-1}$. In particular for every $\varepsilon \in (0, 1)$, we have

$$-\Delta u - (c + \varepsilon V)|x|^{-2}u \geq \frac{1}{2}u^p \quad \text{in } \mathcal{D}'(\mathcal{C}_\Sigma^{1/2}),$$

where $V = \frac{C_0^{p-1}}{2}\chi_{\tilde{\Sigma}}$. We notice that for ε small, $c + \varepsilon V < \mu(\mathcal{C}_\Sigma)$. We apply once more Lemma 2.1 to get

$$(2.4) \quad u(x) \geq C_1 |x|^{\frac{2-N}{2} + \sqrt{(N-2)^2/4 + \lambda_{1,\varepsilon}}} \quad \forall x \in \mathcal{C}_\Sigma^{1/4},$$

where $\lambda_{1,\varepsilon}$ is the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}\Phi - (c + \varepsilon V)\Phi = \lambda_{1,\varepsilon}\Phi$ on Σ . We observe that, for ε small, $\lambda_{1,\varepsilon} < \lambda_1(\Sigma) - c < 0$ and thus

$$p - 1 \geq \frac{2}{\alpha_\Sigma^-} > \frac{2}{\frac{N-2}{2} - \sqrt{(N-2)^2/4 + \lambda_{1,\varepsilon}}} > 0.$$

Recalling that $-\Delta u \geq u^{p-1}u$, we deduce from (2.4) that

$$-\Delta u - \rho(x)|x|^{-2}u \geq 0 \quad \text{in } \mathcal{D}'(\mathcal{C}_{\Sigma}^{1/4}),$$

where $\rho(x) = \frac{C_1^{p-1}}{2}|x|^{(\frac{2-N}{2} + \sqrt{(N-2)^2/4 + \lambda_1, \varepsilon})(p-1)+2}$. Since $\rho(x) \rightarrow +\infty$ as $|x| \rightarrow 0$, applying Lemma 1.4, we contradict the sharpness of the Hardy constant $\mu(\mathcal{C}_{\Sigma})$.

Case 2: $c = \mu(\mathcal{C}_{\Sigma})$.

We consider the function $v \in H(\mathcal{C}_{\Sigma}^1)$ solving

$$-\Delta v - \mu(\mathcal{C}_{\Sigma})|x|^{-2}v = \min(u^p, 1).$$

Then by Lemma 1.3 and the maximum principle $0 < v \leq u$ in \mathcal{C}_{Σ}^1 . By Lemma 2.1,

$$v(x) \geq C|x|^{\frac{2-N}{2}} \quad \text{for } x \in \mathcal{C}_{\Sigma}^{1/2}.$$

Since $-\Delta u - \mu(\mathcal{C}_{\Sigma})|x|^{-2}u = u^{p-1}u$ in $\mathcal{D}'(\mathcal{C}_{\Sigma}^1)$, by Lemma 1.4 and the above estimate, we have

$$\begin{aligned} \infty > \|v\|_{H(\mathcal{C}_{\Sigma}^1)}^2 &\geq \int_{\mathcal{C}_{\Sigma}^1} u^{p-1}v^2 dx \geq \int_{\mathcal{C}_{\Sigma}^1} v^{p+1} dx \\ &\geq C \int_{\mathcal{C}_{\Sigma}^{1/2}} v^{p+1} dx \geq C \int_{\mathcal{C}_{\Sigma}^{1/2}} |x|^{-N} dx = \infty. \end{aligned}$$

This readily leads to a contradiction. Theorem 0.1 is completely proved. \square

3 Smooth domains

In this section, we introduce a system of coordinates near $0 \in \partial\Omega$ that flattens $\partial\Omega$, see [19]. This will allow us to construct a (super-) sub-solution via the function $y^1|y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}}$ which is the (virtual) ground state for the operator $\Delta + c|y|^{-2}$ in the half-space \mathbb{R}_+^N .

3.1 Fermi coordinates

We denote by $\{E_1, E_2, \dots, E_N\}$ the standard orthonormal basis of \mathbb{R}^N and we put

$$\begin{aligned} \mathbb{R}_+^N &= \{y \in \mathbb{R}^N : y^1 > 0\}, \quad \mathbb{S}_+^{N-1} = \mathbb{S}^{N-1} \cap \mathbb{R}_+^N, \\ B_r(y_0) &= \{y \in \mathbb{R}^N : |y - y_0| < r\}, \quad B_r^+ = B_r(0) \cap \mathbb{R}_+^N. \end{aligned}$$

Let \mathcal{U} be an open subset of \mathbb{R}^N with boundary $\mathcal{M} := \partial\mathcal{U}$ a smooth closed hyper-surface of \mathbb{R}^N and $0 \in \mathcal{M}$. We write $N_{\mathcal{M}}$ for the unit normal vector-field of \mathcal{M} pointed into \mathcal{U} . Up to a rotation, we assume that $N_{\mathcal{M}}(0) = E_1$. For $x \in \mathbb{R}^N$, we let $d_{\mathcal{M}}(x) = \text{dist}(\mathcal{M}, x)$ be the distance function of \mathcal{M} . Given $x \in \mathcal{U}$ and close to \mathcal{M} then it can be written uniquely as $x = \sigma_x + d_{\mathcal{M}}(x) N_{\mathcal{M}}(\sigma_x)$, where σ_x is the projection of x on \mathcal{M} . We further use the Fermi coordinates (y^2, \dots, y^N) on \mathcal{M} so that for σ_x close to 0, we have

$$\sigma_x = \text{Exp}_0 \left(\sum_{i=2}^N y^i E_i \right),$$

where $\text{Exp}_0 : \mathbb{R}^{N-1} \rightarrow \mathcal{M}$ is the exponential mapping on \mathcal{M} endowed with the metric induced by \mathbb{R}^N , see [11]. In this way a neighborhood of 0 in \mathcal{U} can be parameterize by the map

$$F_{\mathcal{M}}(y) = \text{Exp}_0 \left(\sum_{i=2}^N y^i E_i \right) + y^1 N_{\mathcal{M}} \left(\text{Exp}_0 \left(\sum_{i=2}^N y^i E_i \right) \right), \quad y \in B_r^+,$$

for some $r > 0$. In this coordinates, the Laplacian Δ is given by

$$\sum_{i=1}^N \frac{\partial^2}{(\partial x^i)^2} = \frac{\partial^2}{(\partial y^1)^2} + h_{\mathcal{M}} \circ F_{\mathcal{M}} \frac{\partial}{\partial y^1} + \sum_{i,j=2}^N \frac{\partial}{\partial y^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial y^j} \right),$$

where $h_{\mathcal{M}}(x) = \Delta d_{\mathcal{M}}(x)$; for $i, j = 2, \dots, N$, $g_{ij} = \langle \frac{\partial F_{\mathcal{M}}}{\partial y^i}, \frac{\partial F_{\mathcal{M}}}{\partial y^j} \rangle$; the quantity $|g|$ is the determinant of g and g^{ij} is the component of the inverse of the matrix $(g_{ij})_{2 \leq i, j \leq N}$.

Since $g_{ij} = \delta_{ij} + O(y^1) + O(|y|^2)$ (see [19]), we have the following Taylor expansion

$$(3.1) \quad \sum_{i=1}^N \frac{\partial^2}{(\partial x^i)^2} = \sum_{i=1}^N \frac{\partial^2}{(\partial y^i)^2} + h_{\mathcal{M}} \circ F_{\mathcal{M}} \frac{\partial}{\partial y^1} + \sum_{i=2}^N O_i(|y|) \frac{\partial}{\partial y^i} + \sum_{i,j=2}^N O_{ij}(|y|) \frac{\partial^2}{\partial y^i \partial y^j}.$$

For $a \in \mathbb{R}$, we put $X_a(t) := |\log t|^a$, $t \in (0, 1)$ and for $c \leq \frac{N^2}{4}$, set

$$\overline{\omega}_a(y) := y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} X_a(|y|) \quad \forall y \in \mathbb{R}_+^N$$

and put

$$L_y := - \sum_{i=1}^N \frac{\partial^2}{(\partial y^i)^2} - c |y|^{-2} + a(a-1) |y|^{-2} X_{-2}(|y|).$$

Then one easily verifies that

$$\begin{cases} L_y \bar{\omega}_a = 2a \sqrt{\frac{N^2}{4} - c} |y|^{-2} X_{-1}(|y|) \bar{\omega}_a & \text{in } \mathbb{R}_+^N, \\ \bar{\omega}_a = 0 & \text{on } \partial \mathbb{R}_+^N \setminus \{0\}, \\ \bar{\omega}_a \in H^1(B_R^+) \quad \forall R > 0, \quad \forall c < \frac{N^2}{4}, \quad \forall a \leq 0. \end{cases}$$

For $K \in \mathbb{R}$, we define

$$\omega_{a,K}(y) = e^{Ky^1} \bar{\omega}_a(y).$$

This function satisfies similar boundary and integrability conditions as $\bar{\omega}_a$. In addition it holds that

(3.2)

$$\begin{aligned} L_y \omega_{a,K} = & -\frac{2K}{y^1} \omega_{a,K} + 2a \sqrt{\frac{N^2}{4} - c} |y|^{-2} X_{-1}(|y|) \bar{\omega}_a \\ & + 2K \left(\frac{N}{2} - \sqrt{\frac{N^2}{4} - c} + a X_{-1}(|y|) \right) \frac{y^1}{|y|^2} \omega_{a,K} - K^2 \omega_{a,K}. \end{aligned}$$

Furthermore for all $a \in \mathbb{R}$

$$\begin{aligned} \sum_{i=2}^N O_i(|y|) \frac{\partial \omega_{a,K}}{\partial y^i} + \sum_{i,j=2}^N O_{ij}(|y|) \frac{\partial^2 \omega_{a,K}}{\partial y^i \partial y^j} &= y^1 e^{Ky^1} O \left(|y|^{-\frac{N}{2}-1+\sqrt{\frac{N^2}{4}-c}} X_a(|y|) \right) \\ &= \mathcal{O}_{a,K}(|y|^{-1}) \omega_{a,K}(y). \end{aligned}$$

Here the error term $\mathcal{O}_{a,K}$ has the property that for any $A > 0$ and $c_0 < \frac{N^2}{4}$, there exit some constants $C > 0$ and $s_0 > 0$ such that

$$(3.3) \quad |\mathcal{O}_{a,K}(s)| \leq C s \quad \forall s \in (0, s_0), \quad \forall a \in [-A, A] \quad \forall c \in \left[c_0, \frac{N^2}{4} \right].$$

Let

$$(3.4) \quad W_{a,K}(x) := \omega_{a,K}(F_{\mathcal{M}}^{-1}(x)), \quad \forall x \in \mathcal{B}_r^+ := F_{\mathcal{M}}(B_r^+).$$

Then using (3.1), (3.2) and the fact that $|x| = |y| + O(|y|^2)$ we obtain the following expansions

(3.5)

$$L_x W_{a,K} = - \left(\frac{2K + h_{\mathcal{M}}(x)}{d_{\mathcal{M}}(x)} \right) W_{a,K} + 2a \sqrt{\frac{N^2}{4} - c} |x|^{-2} X_{-1}(|x|) W_{a,K} + \mathcal{O}_{a,K}(|x|^{-1}) W_{a,K},$$

with $L_x := -\Delta - c|x|^{-2} + a(a-1)|x|^{-2}X_{-2}(|x|)$. Moreover it is easy to see that

$$(3.6) \quad \begin{cases} W_{a,K} > 0 & \text{in } \mathcal{B}_r^+, \\ W_{a,K} = 0 & \text{on } \mathcal{M} \cap \partial\mathcal{B}_r^+ \setminus \{0\}, \\ W_{a,K} \in H^1(\mathcal{B}_r^+) & \forall c < \frac{N^2}{4}, \text{ and } \forall a \leq 0. \end{cases}$$

3.2 Non-existence

We start by recalling the following local improved Hardy inequality. Given a domain $\Omega \subset \mathbb{R}^N$, of class C^2 at $0 \in \partial\Omega$, there exist two constants $C(\Omega) > 0$ and $r_0 = r_0(\Omega) > 0$ such that

$$(3.7) \quad \int_{\Omega_{r_0}} |\nabla u|^2 dx - c \int_{\Omega_{r_0}} |x|^{-2} u^2 dx \geq C(\Omega) \int_{\Omega_{r_0}} u^2 dx \quad \forall u \in C_c^\infty(\Omega_{r_0}),$$

for every $c \in \left(-\infty, \frac{N^2}{4}\right]$, with $\Omega_{r_0} := \Omega \cap B_{r_0}(0)$, see [15]. From this we can define the space $H(\Omega_{r_0})$ to be the completion of $C_c^\infty(\Omega_{r_0})$ with respect to the scalar product

$$\int_{\Omega_{r_0}} \nabla u \nabla v - c \int_{\Omega_{r_0}} |x|^{-2} uv \quad \forall u, v \in C_c^\infty(\Omega_{r_0}).$$

In the sequel we will assume that Ω contains the ball $B = B_1(E_1)$ such that $\partial B \cap \partial\Omega = \{0\}$. Recalling the notations in Section 3.1, we state the following result

Lemma 3.1 *Let $c_0 \in \left(-\infty, \frac{N^2}{4}\right]$ and $f \in L^\infty(\Omega_{r_0})$ be a non-negative and non-trivial function. For $c \in \left[c_0, \frac{N^2}{4}\right]$, let $v \in H(\Omega_{r_0})$ be the unique solution of the problem*

$$\int_{\Omega_{r_0}} \nabla v \nabla \phi dx - c \int_{\Omega_{r_0}} |x|^{-2} v \phi dx = \int_{\Omega_{r_0}} f \phi dx \quad \forall \phi \in C_c^\infty(\Omega_{r_0}).$$

Then there exist $R > 0$ and $r > 0$ such that

$$v(F_{\partial B}(y)) \geq R y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} \quad \forall y \in B_r^+ \quad \forall c \in \left[c_0, \frac{N^2}{4}\right].$$

Proof. For $a \leq 0$ and $r > 0$ small we define $G_r^+ := F_{\partial B}(B_r^+)$ and

$$w_a(x) := \omega_{a,N-1}(F_{\partial B}^{-1}(x)), \quad \forall x \in G_r^+.$$

Letting $L := -\Delta - c|x|^{-2}$, by (3.5),

$$\begin{aligned} L(w_0 + w_{-1}) &\leq -\frac{2(N-1) + h_{\partial B}}{d_{\partial B}}(w_0 + w_{-1}) - 2|x|^{-2}X_{-2}(|x|)w_{-1} \\ &\quad - 2\sqrt{\frac{N^2}{4} - c}|x|^{-2}X_{-1}(|x|)w_{-1} + O(|x|^{-1})(w_0 + w_{-1}). \end{aligned}$$

We observe that

$$(3.8) \quad w_{-1}(x) = w_0(x)|\log|F_{\partial B}^{-1}(x)||^{-1} = w_0(x)(X_{-1}(|x|) + O(|x|)).$$

Since $-h_{\partial B}(x) = (N-1)(1 + O(|x|))$ in G_r^+ , we have using (3.8),

$$(3.9) \quad L(w_0 + w_{-1}) \leq 0 \quad \text{in } G_r^+,$$

for r positive small.

Case $c \in \left[c_0, \frac{N^2}{4}\right)$.

We put $U = w_0 + w_{-1}$. Then $U \in H^1(G_r^+) \cap C(G_r^+)$ by (3.6) and $v \in H_0^1(\Omega_{r_0}) \cap C(\Omega_{r_0})$ by elliptic regularity theory and Remark 1.1. Moreover $v > 0$ in Ω_{r_0} by the maximum principle. Therefore since $F_{\partial B}(\overline{r\mathbb{S}_+^{N-1}}) \subset \Omega_{r_0}$, we can let

$$(3.10) \quad R = r^{\frac{N-2}{2}} e^{-(N-1)r} \inf_{y \in r\mathbb{S}_+^{N-1}} v > 0$$

so that

$$RU \leq v \quad \text{on } F_{\partial B}(\overline{r\mathbb{S}_+^{N-1}}).$$

By (3.6) and setting $\varphi = RU - v$, we get $\varphi^+ := \max(\varphi, 0) \in H_0^1(G_r^+)$ because $U = 0$ on $\partial B \cap \partial G_r^+$. Since $Lv \geq 0$, we have

$$L\varphi \leq 0 \quad \text{in } G_r^+,$$

by (3.9). Multiplying the above inequality by φ^+ and integrating by parts yields

$$\int_{G_r^+} |\nabla \varphi^+|^2 dx - c \int_{G_r^+} |x|^{-2} |\varphi^+|^2 dx \leq 0.$$

This implies that $\varphi^+ \equiv 0$ in G_r^+ for all r positive small. We conclude that $v \geq R(w_0 + w_{-1})$ in G_r^+ and thus

$$v(F_{\partial B}(y)) \geq R w_0(F_{\partial B}(y)) \geq R y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}}, \quad \forall y \in G_r^+, \quad \forall c \in \left[c_0, \frac{N^2}{4}\right).$$

Case $c = \frac{N^2}{4}$.

In this case, we notice that the solutions v_k to the problem

$$\int_{\Omega_{r_0}} \nabla v_k \nabla \phi \, dx - \left(\frac{N^2}{4} - \frac{1}{k} \right) \int_{\Omega_{r_0}} |x|^{-2} v_k \phi \, dx = \int_{\Omega_{r_0}} f \phi \, dx \quad \forall \phi \in H(\Omega_{r_0})$$

are H_0^1 -solutions if r_0 is small enough (independent on k) and they are monotone increasing to v as $k \rightarrow \infty$. Hence by (3.10) and from the above argument we deduce that there exist an integer $k_0 \geq 1$ and a constant R (possibly depending on k_0) such that

$$v(F_{\partial B}(y)) \geq v_k(F_{\partial B}(y)) \geq R y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{1}{k}}}, \quad \forall y \in G_r^+, \quad \forall k \geq k_0.$$

Passing to the limit as $k \rightarrow \infty$, we get the result. \square

3.2.1 Proof of Theorem 0.2

Recall that

$$(3.11) \quad \alpha_{\mathbb{S}_+^{N-1}}^- = \frac{N-2}{2} - \sqrt{\frac{N^2}{4} - c}, \quad p \geq p_{\mathbb{S}_+^{N-1}} := 1 + \frac{2}{\alpha_{\mathbb{S}_+^{N-1}}^-},$$

Suppose that $u \neq 0$ near 0 thus we can find a bounded function f with $f \neq 0$ and $0 \leq f \leq u^p$. By Lemma 1.3 and the maximum principle, there exists $v \in H(\Omega_{r_0})$ such that $u \geq v > 0$ and

$$\int_{\Omega_{r_0}} \nabla v \nabla \phi \, dx - c \int_{\Omega_{r_0}} |x|^{-2} v \phi \, dx = \int_{\Omega_{r_0}} f \phi \, dx \quad \forall \phi \in C_c^\infty(\Omega_{r_0}),$$

for some $r_0 > 0$ small. In addition Lemma 3.1 yields

$$(3.12) \quad v(F_{\partial B}(y)) \geq R y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} \quad \forall y \in B_r^+.$$

Case 1: $c \in \left(N-1, \frac{N^2}{4} \right)$.

Since $-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c} < 0$, (3.12) implies that

$$(3.13) \quad u(x) \geq v(x) \geq C d_{\partial B}(x) |x|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} \quad \forall x \in G_r^+,$$

where we recall that $d_{\partial B}(F_{\partial B}(y)) = y^1$ and $|F_{\partial B}^{-1}(x)| \leq C|x|$.

Let $\gamma \in (0, 1)$ then for every $x \in B_\gamma(\gamma E_1) \subset B_1(E_1) = B$, we have

$$d_{\partial B}(x) = 1 - |x - E_1| > (1 - \gamma)x^1.$$

Using this together with (3.11) and (3.13), we obtain

$$(3.14) \quad u^{p-1}(x) \geq C \left(\frac{x^1}{|x|} \right)^{p-1} |x|^{-2} \quad \forall x \in G_r^+.$$

Since for $\gamma > 0$ small u satisfies

$$-\Delta u - c|x|^{-2}u \geq \frac{1}{2}u^{p-1}u + \frac{1}{2}u^p \quad \text{in } \mathcal{D}'(B_\gamma(\gamma E_1)),$$

we thus have from (3.14)

$$-\Delta u - V_\varepsilon \left(\frac{x}{|x|} \right) |x|^{-2}u \geq \frac{1}{2}u^p \quad \text{in } \mathcal{D}'(B_\gamma(\gamma E_1)),$$

where $V_\varepsilon \left(\frac{x}{|x|} \right) = c + \frac{\varepsilon}{2}C \left(\frac{x}{|x|} \cdot E_1 \right)^{p-1}$ for every $\varepsilon \in (0, 1)$. From now on, we will fix ε so small that $V_\varepsilon < \frac{N^2}{4}$.

Given $\delta \in (0, 1)$, consider the cone $\mathcal{C}_\delta := \{x \in \mathbb{R}^N : x^1 > \delta|x|\}$ and define $\Sigma_\delta = \mathcal{C}_\delta \cap \mathbb{S}^{N-1}$. We observe that for every $\delta \in (0, 1)$, there exists $r_\delta > 0$ such that the cone-like domain

$$\mathcal{C}_{\Sigma_\delta}^{r_\delta} \subset B_\gamma(\gamma E_1).$$

It follows that

$$-\Delta u - V_\varepsilon \left(\frac{x}{|x|} \right) |x|^{-2}u \geq \frac{u^p}{2} \quad \text{in } \mathcal{D}'(\mathcal{C}_{\Sigma_\delta}^{r_\delta}).$$

Let $\lambda_{1,\delta,\varepsilon}$ be the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}\Phi - V_\varepsilon\Phi = \lambda_{1,\delta,\varepsilon}\Phi$ on Σ_δ . Since $\lambda_1(\Sigma_\delta) \searrow N-1 = \lambda_1(\mathbb{S}_+^{N-1})$ as $\delta \rightarrow 0$, we can choose a $\delta_\varepsilon \in (0, 1)$ such that

$$(3.15) \quad \lambda_{1,\delta,\varepsilon} < N-1-c < 0 \quad \forall \delta \in (0, \delta_\varepsilon).$$

Since $V_\varepsilon \leq \frac{N^2}{4} < \frac{(N-2)^2}{4} + \lambda_1(\Sigma_\delta) = \mu(\mathcal{C}_{\Sigma_\delta})$ for every $\delta \in (0, \delta_\varepsilon)$, we can apply Lemma 2.1 to have $\forall \delta \in (0, \delta_\varepsilon)$

$$u(x) \geq C|x|^{\frac{2-N}{2} + \sqrt{\frac{(2-N)^2}{4} + \lambda_{1,\delta,\varepsilon}}}, \quad \text{in } \mathcal{C}_{\Sigma_\delta}^{r_\delta/2}$$

where $\tilde{\Sigma}_\delta \subset\subset \Sigma_\delta$. We get from (3.15)

$$p-1 \geq \frac{2}{\alpha_{\mathbb{S}_+^{N-1}}^-} > \frac{2}{\frac{N-2}{2} - \sqrt{(N-2)^2/4 + \lambda_{1,\delta,\varepsilon}}} > 0.$$

Since $-\Delta u \geq u^{p-1}u$, we deduce that $\forall \delta \in (0, \delta_\varepsilon)$

$$-\Delta u - \rho(x)|x|^{-2}u \geq 0 \quad \text{in } \mathcal{D}'(\mathcal{C}_{\tilde{\Sigma}_\delta}^{r_\delta/2}),$$

where $\rho(x) \geq C|x|^{(\frac{2-N}{2} + \sqrt{(N-2)^2/4 + \lambda_{1,\delta,\varepsilon}})(p-1)+2}$. Since $\rho(x) \rightarrow +\infty$ as $|x| \rightarrow 0$, applying Lemma 1.4, we contradict the sharpness of the Hardy constant $\mu(\mathcal{C}_{\tilde{\Sigma}_\delta})$.

Case 2: $c = \frac{N^2}{4}$.

Here, we recall that $p \geq \frac{N+2}{N-2}$. By (3.7) we can let $\zeta \in H(G_r^+)$ be the unique solution to the problem

$$\int_{G_r^+} \nabla \zeta \nabla \phi \, dx - \frac{N^2}{4} \int_{G_r^+} |x|^{-2} \zeta \phi \, dx = \int_{G_r^+} 1 \phi \, dx \quad \forall \phi \in C_c^\infty(G_r^+).$$

We put $\Phi(y) = \frac{y^1}{|y|}$. Then by Lemma 1.4, Lemma 3.1 and (3.12), we have

$$\begin{aligned} \|\zeta\|_{H(G_r^+)}^2 &\geq \int_{G_r^+} v^{p-1} |\zeta|^2 \, dx \\ &\geq C \int_{B_r^+} |y|^{\frac{2-N}{2}(p+1)} \Phi^{p+1} \sqrt{|\hat{g}|}(y) \, dy \\ &\geq C \int_{B_r^+} |y|^{\frac{2-N}{2}(p+1)} \Phi^{p+1} \, dy \\ &= C \int_{\mathbb{S}_+^{N-1}} \Phi^{p+1} \, d\sigma \int_0^r t^{\frac{2-N}{2}(p+1)} t^{N-1} \, dt \\ &\geq C \int_{\mathbb{S}_+^{N-1}} \Phi^{p+1} \, d\sigma \int_0^r t^{-1} \, dt = \infty. \end{aligned}$$

This clearly contradicts the fact that $\zeta \in H(G_r^+)$. □

3.3 Existence

Let Ω be a domain of \mathbb{R}^N , $N \geq 3$ which is of class C^2 at $0 \in \Omega$, we shall show that for some $r > 0$ small, there exists a positive function $u \in L^p(\Omega \cap B_r(0))$,

$1 < p < p_{\mathbb{S}_+^{N-1}} = 1 + \frac{2}{\alpha_{\mathbb{S}_+^{N-1}}^-}$ and

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega \cap B_r(0)).$$

Letting B be a unit ball with $0 \in \partial B$, call $\mathcal{U} = \mathbb{R}^N \setminus \overline{B}$ and $\mathcal{M} = \partial\mathcal{U}$. Under the notations in Section 3.1, the above existence result is a consequence of the following

Proposition 3.2 *Let $1 < p < p_{\mathbb{S}_+^{N-1}}$ and $N - 1 < c \leq \frac{N^2}{4}$. Then there exists $r > 0$ small such that the problem*

$$(3.16) \quad \begin{cases} -\Delta w - \frac{c}{|x|^2} w = w^p & \text{in } \mathcal{D}'(\mathcal{B}_r^+), \\ w \in L^p(\mathcal{B}_r^+), \\ w > 0 & \mathcal{B}_r^+ \end{cases}$$

has a supersolution, with $\mathcal{B}_r^+ = F_{\mathcal{M}}(B_r^+)$.

Proof. Notice that $h_{\mathcal{M}}(x) = \frac{N-1}{1+d_{\mathcal{M}}(x)}$ and thus

$$(3.17) \quad -\frac{2(1-N) + h_{\mathcal{M}}(x)}{d_{\mathcal{M}}(x)} \geq \frac{N-1}{d_{\mathcal{M}}(x)} \quad \forall x \in \mathcal{U}.$$

Define (see (3.4))

$$w(x) = \omega_{\frac{1}{2p}, 1-N}(F_{\mathcal{M}}^{-1}(x)) \quad \forall x \in \mathcal{B}_r^+.$$

By (3.5), (3.17) and using the fact that $|x| = |y| + O(|y|^2)$, we have

$$\begin{aligned} -\Delta w(F_{\mathcal{M}}(y)) - c|F_{\mathcal{M}}(y)|^{-2} w(F_{\mathcal{M}}(y)) &\geq \frac{2p-1}{4p^2} |y|^{-2} X_{-2}(|y|) w(F_{\mathcal{M}}(y)) \\ &\quad + O(|y|^{-1}) w(F_{\mathcal{M}}(y)). \end{aligned}$$

In particular if $r > 0$ small

$$-\Delta w(F_{\mathcal{M}}(y)) - c|F_{\mathcal{M}}(y)|^{-2} w(F_{\mathcal{M}}(y)) \geq C|y|^{-2} X_{-2}(|y|) w(F_{\mathcal{M}}(y)) \quad \forall y \in B_r^+,$$

with $C > 0$ a constant depending only on p and N . Therefore w is a supersolution provided

$$C|y|^{-2} X_{-2}(|y|) w(F_{\mathcal{M}}(y)) \geq w(F_{\mathcal{M}}(y))^p \quad \forall y \in B_r^+$$

or equivalently

$$C \frac{y^1}{|y|} e^{(1-N)y^1} |y|^{-\alpha_{\mathbb{S}_+^{N-1}}^- - 2} |\log |y||^{-2+\frac{1}{2p}} \geq \left(\frac{y^1}{|y|} e^{(1-N)y^1} |y|^{-\alpha_{\mathbb{S}_+^{N-1}}^-} |\log |y||^{\frac{1}{2p}} \right)^p \quad \forall y \in B_r^+.$$

Since $0 < \frac{y^1}{|y|} e^{(1-N)y^1} \leq 1$ and $p \geq 1$, the above holds if

$$C |y|^{-\alpha_{\mathbb{S}_+^{N-1}}^- - 2} |\log |y||^{-2+\frac{1}{2p}} \geq \left(|y|^{-\alpha_{\mathbb{S}_+^{N-1}}^-} |\log |y||^{\frac{1}{2p}} \right)^p \quad \forall y \in B_r^+.$$

The previous inequality is true provided

$$C |y|^{-\alpha_{\mathbb{S}_+^{N-1}}^- - 2 + p\alpha_{\mathbb{S}_+^{N-1}}^-} |\log |y||^{-2+\frac{1}{2p}-\frac{1}{2}} \geq 1 \quad \forall y \in B_r^+.$$

This is clearly possible whenever $p < p_{\mathbb{S}_+^{N-1}} = 1 + \frac{2}{\alpha_{\mathbb{S}_+^{N-1}}^-}$ and $r > 0$ is small enough.

Finally, we notice that $\int_{B_r^+} w^p dx \leq C \int_0^r t^{\frac{N-4}{2}-\sqrt{\frac{N^2}{4}-c}} |\log t|^{\frac{1}{2}} dt < \infty$, when $N-1 < c \leq \frac{N^2}{4}$. This concludes the proof. \square

4 Problem with perturbation

We let $\Gamma \subset \mathbb{R}^N$ be a smooth closed submanifold of dimension k with $1 \leq k < N-2$.

Let Ω be a smooth domain in \mathbb{R}^N containing Γ . We study the problem

$$(4.1) \quad \begin{cases} -\Delta u - \frac{(N-k-2)^2}{4} \frac{1}{\text{dist}(x, \Gamma)^2} q(x) u \geq u^p & \text{in } \mathcal{D}'(\Omega \setminus \Gamma), \\ u \in L_{loc}^p(\Omega \setminus \Gamma), \\ u \not\geq 0 & \text{in } \Omega \setminus \Gamma, \end{cases}$$

where $q \in C^2(\Omega)$, $q \geq 0$ in Ω and normalized as

$$(4.2) \quad \max_{\sigma \in \Gamma} q(\sigma) = 1.$$

We obtain the following result:

Theorem 4.1 *Suppose that $p \geq \frac{N-k+2}{N-k-2}$ and that (4.2) holds. Then problem (4.1) does not have a solution.*

The above supercriticality assumption on p is sharp as we will see in Section 4.2 below.

Remark 4.2 • *It was observed in [[5], Remark 3] that if $0 < \max_{\Gamma} q < 1$ or $q \equiv 1$ then (4.1) does not have a solution when*

$$p \geq p^+ := 1 + \frac{2}{\frac{N-k-2}{2} - \sqrt{\frac{(N-k-2)^2}{4} - c}} \geq \frac{N-k+2}{N-k-2},$$

with $c = \frac{(N-k-2)^2}{4} \max_{\Gamma} q$.

- *We should mention that extremals for weighted Hardy inequality was studied in [6], [7] and [14] when Γ is a submanifold of $\partial\Omega$ and $k = 1, \dots, N-1$. In these papers, the finiteness of the integral $\int_{\Gamma} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma$ was necessary and sufficient to obtain the existence of an eigenfunction in some function space corresponding to some “critical” eigenvalue.*

We believe that the argument in this paper and the results in [14] might be used to study problem (4.1) but with $\Gamma \subset \partial\Omega$.

In the sequel, we denote by $\delta(x) := \text{dist}(x, \Gamma)$. For $\beta > 0$, we consider the interior of the tube around Γ of radius β defined as $\Gamma_{\beta} := \{x \in \Omega : \delta(x) < \beta\}$. It is well known that if β is positive small, the function δ is smooth in $\Gamma_{\beta} \setminus \Gamma$. If β is small then for all $x \in \Gamma_{\beta}$, there exists a unique projection $\sigma(x) \in \Gamma$ given by

$$(4.3) \quad \sigma(x) = x - \frac{1}{2} \nabla(\delta^2)(x) = x - \delta(x) \nabla \delta(x).$$

In addition the function σ is also smooth in Γ_{β} , see for instance [1].

From now on, we will consider β 's for which the projection function σ is smooth. Set

$$(4.4) \quad \omega_0(x) = \delta^{-\alpha(x)},$$

with

$$(4.5) \quad \alpha(x) = \alpha_q(x) = \frac{N-k-2}{2} - \sqrt{\tilde{\alpha}(x)}$$

and where

$$\tilde{\alpha}(x) = \left(\frac{N-k-2}{2} \right)^2 (1 - q(\sigma(x)) + \delta(x)).$$

Clearly α is well defined as soon as $q \leq 1$ on Γ . Recall that $X_a(t) = |\log t|^a$, $t \in (0, 1)$ and $a \in \mathbb{R}$. We define

$$\omega_a(x) := \omega_0(x) X_a(\delta(x)).$$

We will need the following result which will be useful in the proof of Theorem 4.1.

Lemma 4.3 *Put $L_q := -\Delta - \left(\frac{N-k-2}{2}\right)^2 \delta^{-2} q$. Then there exist $C, \beta_0 > 0$ depending only on Γ , a and $\|q\|_{C^2(\overline{\Omega})}$ such that*

$$(4.6) \quad \left| L_q \omega_a - 2a \sqrt{\tilde{\alpha}} \delta^{-2} X_{-1} \omega_a + a(a-1) \delta^{-2} X_{-2} \omega_a \right| \leq C |\log(\delta)| \delta^{-\frac{3}{2}} \omega_a, \quad \text{in } \Gamma_{\beta_0}.$$

Proof. We start by noticing that

$$(4.7) \quad \Delta \omega_0 = \omega_0 \left(\Delta \log(\omega_0) + |\nabla \log(\omega_0)|^2 \right)$$

and that

$$(4.8) \quad -\Delta \log(\omega_0) = \Delta \alpha \log(\delta) + 2 \nabla \alpha \cdot \nabla(\log(\delta)) + \alpha \Delta \log(\delta).$$

We have

$$(4.9) \quad -\Delta \alpha = \Delta \sqrt{\tilde{\alpha}} = \sqrt{\tilde{\alpha}} \left(\frac{1}{2} \Delta \log(\tilde{\alpha}) + \frac{1}{4} |\nabla \log(\tilde{\alpha})|^2 \right).$$

By simple computations we get

$$\sqrt{\tilde{\alpha}} \nabla \log(\tilde{\alpha}) = \frac{\nabla \tilde{\alpha}}{\sqrt{\tilde{\alpha}}} = \left(\frac{N-k-2}{2} \right)^2 \frac{-\nabla(q \circ \sigma) + \nabla \delta}{\sqrt{\tilde{\alpha}}}$$

and

$$\sqrt{\tilde{\alpha}} |\Delta \log(\tilde{\alpha})| \leq \frac{|\Delta \tilde{\alpha}|}{\sqrt{\tilde{\alpha}}} + \frac{|\nabla \tilde{\alpha}|^2}{\sqrt{\tilde{\alpha}}}.$$

We deduce that there exists a constant $\beta_0 > 0$ depending only on Γ and $\|q\|_{C^2(\overline{\Omega})}$ such that

$$(4.10) \quad |\Delta \alpha| \leq \frac{C}{\delta^{\frac{3}{2}}} \quad \text{in } \Gamma_{\beta_0}.$$

Similar we have

$$(4.11) \quad |\nabla \alpha \cdot \nabla \log \delta| \leq \frac{C}{\delta^{\frac{3}{2}}} \quad \text{in } \Gamma_{\beta_0}.$$

Recall that (see for instance [9])

$$(4.12) \quad \alpha \Delta \log(\delta) = \alpha \frac{N-k-2}{\delta^2} (1 + O(\delta)).$$

Using (4.9), (4.10), (4.11) and (4.12) in the formula (4.8), we obtain the following estimate:

$$(4.13) \quad \left| \Delta \log(\omega_0) + \alpha \frac{N-k-2}{\delta^2} \right| \leq C \frac{|\log \delta|}{\delta^{\frac{3}{2}}} \quad \text{in } \Gamma_{\beta_0}.$$

We also have

$$-\nabla(\log(\omega_0)) = \nabla(\alpha \log(\delta)) = \alpha \frac{\nabla \delta}{\delta} + \log(\delta) \nabla \alpha$$

and thus

$$(4.14) \quad \left| |\nabla(\log(\omega_0))|^2 - \frac{\alpha^2}{\delta^2} \right| \leq C \frac{|\log \delta|}{\delta^{\frac{3}{2}}} \quad \text{in } \Gamma_{\beta_0}.$$

By using (4.13), (4.14) in the identity (4.7), we conclude that

$$\left| \frac{\Delta \omega_0}{\omega_0} + \alpha \frac{N-k-2}{\delta^2} - \frac{\alpha^2}{\delta^2} \right| \leq C \frac{|\log \delta|}{\delta^{\frac{3}{2}}} \quad \text{in } \Gamma_{\beta_0}.$$

We use the fact that $|q(x) - q(\sigma(x))| \leq C\delta(x)$ to deduce that

$$\Delta \omega_0 = \frac{(N-k-2)^2}{4} \delta^{-2} q(x) \omega_0 + O(|\log(\delta)| \delta^{-\frac{3}{2}}) \omega_0 \quad \text{in } \Gamma_{\beta_0}.$$

To conclude, we write

$$\omega_a(x) := \omega_0(x) (-\log(\delta(x)))^a$$

and the proof of (4.6) follows with some little computations. We skip the details. \square

4.1 Proof of Theorem 4.1

Step I: The following inequality holds:

$$(4.15) \quad \int_{\Gamma_{\beta_0}} |\nabla \varphi|^2 dx - \frac{(N-k-2)^2}{4} \int_{\Gamma_{\beta_0}} \delta^{-2} q \varphi^2 dx \geq C \int_{\Gamma_{\beta_0}} \delta^{-2} X_{-2} \varphi^2 dx$$

for any $\varphi \in C_c^\infty(\Gamma_{\beta_0})$, with $\beta_0 > 0$ small depending only on K and $\|q\|_{C^2(\overline{\Omega})}$ and $C > 0$ is a constant.

Indeed, observe that by (4.6),

$$-\frac{\Delta\omega_{\frac{1}{2}}}{\omega_{\frac{1}{2}}} - \frac{(N-k-2)^2}{4}\delta^{-2}q \geq \frac{1}{4}\delta^{-2}X_{-2} - C|\log(\delta)|\delta^{-\frac{3}{2}} \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

Hence, there exist $\beta_0 > 0$ small and a constant $C > 0$ such that

$$(4.16) \quad -\Delta\omega_{\frac{1}{2}} - \frac{(N-k-2)^2}{4}\delta^{-2}q\omega_{\frac{1}{2}} - C\delta^{-2}X_{-2}\omega_{\frac{1}{2}} \geq 0 \quad \text{in } \Gamma_{\beta_0} \setminus \Gamma.$$

Since $\omega_{\frac{1}{2}} \in L^1(\Gamma_{\beta_0})$, the inequality (4.16) holds in $\mathcal{D}'(\Gamma_{\beta_0})$ thus by Lemma 1.4, (4.15) follows.

Step II: Set $\theta_a := \omega_0 + \omega_a$, with $a < -1/2$. There exist positive constants C and β_0 depending only on a , Γ and $\|q\|_{C^2(\overline{\Omega})}$ such that

$$(4.17) \quad \|\theta_a\|_{H^1(\Gamma_{\beta_0})}^2 \leq C \int_{\Gamma} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma.,$$

First of all it is easy to see that, since $X_a \leq 1$ for a negative, we can estimate

$$(4.18) \quad |\nabla\theta_a|^2 \leq C\delta^{-2\alpha-2} \quad \text{in } \Gamma_{\beta_0}.$$

Following [10], there exists a family of disjoint open sets W_i , $i = 1, \dots, m_0$ of Γ such that

$$\Gamma = \bigcup_{i=1}^{m_0} \overline{W}_i, \quad |\overline{W}_i \cap \overline{W}_j| = 0, \quad i \neq j.$$

Moreover by (4.18),

$$(4.19) \quad \|\theta_a\|_{H^1(\Gamma_{\beta_0})}^2 \leq C \int_{\Gamma_{\beta_0}} \delta^{-2\alpha-2} = C \sum_{i=1}^{m_0} \int_{W_i \times B_{\beta_0}^{N-k}} \delta^{-2\alpha-2} (1 + O_i(\delta)) d\delta d\sigma,$$

where B_{β}^{N-k} is the ball of \mathbb{R}^{N-k} with radius β . Therefore, we have

$$\begin{aligned} \|\theta_a\|_{H^1(\Gamma_{\beta_0})}^2 &\leq C \sum_{i=1}^{m_0} \int_{W_i} \int_{\mathbb{S}^{N-k-1}} \int_0^{\beta_0} \delta^{-1} \delta^{(N-k-2)\sqrt{1-q(\sigma)+\delta}} d\delta d\sigma \\ &\leq C \sum_{i=1}^{m_0} \int_{W_i} \int_{\mathbb{S}^{N-k-1}} \int_0^{\beta_0} \delta^{-1} \delta^{(N-k-2)\sqrt{1-q(\sigma)}} d\delta d\sigma \\ &\leq C \sum_{i=1}^{m_0} \int_{W_i} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma = C \int_{\Gamma} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma. \end{aligned}$$

This ends the proof of this step.

Step III: Let u satisfies (4.1) and $\theta_a = \omega_0 + \omega_a$, for $a < -1/2$. For any $\beta > 0$ small,

there exists a constant $C > 0$ such that

$$(4.20) \quad u \geq C\theta_a \quad \text{in } \Gamma_\beta.$$

Indeed, define $q_n(x) := q(x) - \frac{1}{n}$ with $n \in \mathbb{N}^*$ and we put $\theta_{a,n} = \delta^{-\alpha_{qn}} + \delta^{-\alpha_{qn}} X_a(\delta)$. Recalling (4.5), by (4.6) there exist constants $\beta_0, C > 0$ (independent on n) such that

$$L_{q_n} \theta_{a,n} \leq -\frac{3}{4} \delta^{-2} |\log \delta|^{-2+a} \delta^{-\alpha_{qn}} + C |\log(\delta)| \delta^{-\frac{3}{2}} \delta^{-\alpha_{qn}} \quad \text{in } \Gamma_\beta,$$

for any $\beta \in (0, \beta_0)$. Therefore for all $\beta > 0$ small we obtain

$$(4.21) \quad -\Delta \theta_{a,n} - \frac{(N-k-2)^2}{4} \delta^{-2} q_n(x) \theta_{a,n} \leq 0 \quad \text{in } \Gamma_\beta \quad \forall n \geq 1.$$

By [[5], Lemma 1], $u \in L_{loc}^p(\Omega)$. In addition, it is nonnegative and non-trivial in Ω and satisfies

$$(4.22) \quad -\Delta u - \frac{(N-k-2)^2}{4} \delta^{-2} q(x) u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

Hence by the maximum principle, $u > 0$ in Ω . For $\beta > 0$ small (independent on n), by (4.15) we can pick $v_n \in H_0^1(\Gamma_\beta)$ solution to

$$(4.23) \quad -\Delta v_n - \frac{(N-k-2)^2}{4} \delta^{-2} q_n(x) v_n = \min(u^p, 1) \quad \text{in } \Gamma_\beta.$$

By Lemma 1.3 the sequence $(v_n)_n$ is monotone increasing and converging pointwise to $v \in H(\Gamma_\beta)$ solution to $-\Delta v - \frac{(N-k-2)^2}{4} \delta^{-2} q(x) v = \min(u^p, 1)$. By Lemma 1.3 we have that $u \geq v \geq v_n > 0$ in Γ_β for any $n \geq 1$. By elliptic regularity theory v_n is continuous in $\Gamma_\beta \setminus \Gamma$. We choose $M_n > 0$ such that

$$(4.24) \quad M_n \sup_{\partial \Gamma_{\frac{\beta}{2}}} \theta_a = \inf_{\partial \Gamma_{\frac{\beta}{2}}} v_n.$$

Clearly, we have $M_n \theta_{a,n} \leq v_n$ on $\partial \Gamma_{\frac{\beta}{2}}$. It follows from (4.17) that $(M_n \theta_{a,n} - v_n)^+ \in H_0^1(\Gamma_{\frac{\beta}{2}})$. On the other hand by (4.21) and (4.23),

$$-\Delta (M_n \theta_{a,n} - v_n) - \frac{(N-k-2)^2}{4} \delta^{-2} q(x) (M_n \theta_{a,n} - v_n) \leq 0 \quad \text{in } \Gamma_{\frac{\beta}{2}}.$$

Multiplying this inequality by $(M_n \theta_{a,n} - v_n)^+$ and integrating by parts yields $M_n \theta_{a,n} \leq v_n$ on $\Gamma_{\frac{\beta}{2}}$ by (4.15). Since v_n is monotone increasing to v , by the choice of M_n in (4.24), there exists an integer $n_0 \geq 1$ such that $M_{n_0} \theta_{a,n} \leq v_n$ for all $n \geq n_0$. Passing to the limit, we get (4.20).

Step IV: There is no u satisfying (4.1) with $p \geq \frac{N-k+2}{N-k-2}$.

By using (4.20) we have that

$$u^{p-1} \geq C\theta_a^{p-1} \geq C\omega_0^{p-1} \geq C\delta^{-2+2\sqrt{1-q\sigma}} \quad \text{in } \Gamma_\beta,$$

for some $C > 0$ and provided β is small. This together with (4.22) give

$$(4.25) \quad -\Delta u - \left(q + C_0\delta^{2\sqrt{1-q\sigma}}\right) \frac{(N-k-2)^2}{4} \delta^{-2} u \geq 0 \quad \text{in } \mathcal{D}'(\Gamma_\beta),$$

for some $C_0 > 0$. By Lemma 1.4 we have, $\forall \varphi \in C_c^\infty(\Gamma_\beta)$

$$(4.26) \quad \frac{(N-k-2)^2}{4} \leq \frac{\int_{\Gamma_\beta} |\nabla \varphi|^2 dx}{\int_{\Gamma_\beta} \left(q + C_0\delta^{2\sqrt{1-q(\sigma)}}\right) \delta^{-2} \varphi^2 dx}.$$

Our aim is to construct appropriate test functions in (4.26) supported in a neighborhood of the maximum point of q on Γ in order to get a contradiction.

By (4.2), we can let $\sigma_0 \in \Gamma$ be such that

$$(4.27) \quad q(\sigma_0) = \max_{\sigma \in \Gamma} q(\sigma) = 1.$$

For $y \in \mathbb{R}^N$, we write $y = (\tilde{y}, \bar{y}) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$ with $\tilde{y} = (y^1, \dots, y^{N-k})$ and $\bar{y} = (y^{N-k+1}, \dots, y^N)$. Consider $f : \mathbb{R}^k \rightarrow \Gamma$ a normal parameterization of a neighborhood of σ_0 with $f(0) = \sigma_0$. In a neighborhood of σ_0 , we consider \mathcal{N}_i , $i = 1, \dots, N-k$ an orthonormal frame field on the normal bundle of Γ . We can therefore define a parameterization of a neighborhood, in \mathbb{R}^N , of σ_0 by the mapping $Y : B_r(0) \rightarrow \Gamma_\beta$ as

$$y \mapsto Y(y) = f(\bar{y}) + \sum_{i=1}^{N-k} y^i \mathcal{N}_i(f(\bar{y})) \in \Gamma_\beta,$$

for some $r > 0$ small. By identification using (4.3), we get for some $r > 0$ small

$$(4.28) \quad \delta(Y(y)) = |\tilde{y}|, \quad \sigma(Y(y)) = f(\bar{y}) \quad \forall y \in B_r(0).$$

Denoting by g the metric induced by Y with component $g_{ij}(y) = \langle \partial_i Y(y), \partial_j Y(y) \rangle$, it is not difficult to verify that for all $y \in B_r(0)$

$$(4.29) \quad g_{ij}(y) = \delta_{ij} + O(|y|) \quad \text{for } i, j = 1, \dots, N.$$

Next we let $w \in C_c^\infty(\mathbb{R}^{N-k} \setminus \{0\} \times \mathbb{R}^k)$. We choose $\varepsilon_0 > 0$ small such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\varepsilon \text{Supp} w \subset B_r(0).$$

We define the following test function

$$\varphi_\varepsilon(x) = \varepsilon^{\frac{2-N}{2}} w(\varepsilon^{-1} Y^{-1}(x)), \quad x \in Y(\varepsilon \text{Supp} w).$$

Clearly, for every $\varepsilon \in (0, \varepsilon_0)$, we have that $\varphi_\varepsilon \in C_c^\infty(\Gamma_\beta)$ and thus by (4.26), we have (summations over repeated indices is understood)

$$\begin{aligned} \frac{(N-k-2)^2}{4} &\leq \frac{\int_{\Gamma_\beta} |\nabla \varphi_\varepsilon|^2 dx}{\int_{\Gamma_\beta} \left(q + C_0 \delta^2 \sqrt{1-q(\sigma)} \right) \delta^{-2} \varphi_\varepsilon^2 dx} \\ &= \frac{\varepsilon^{2-N} \int_{\mathbb{R}^N} \varepsilon^{-2} (g^\varepsilon)^{ij} \partial_i w \partial_j w \sqrt{|g^\varepsilon|} dy}{\varepsilon^{2-N} \int_{\mathbb{R}^N} \left(q(Y(\varepsilon y)) + C_0 |\varepsilon \tilde{y}|^2 \sqrt{1-q(f(\varepsilon \bar{y}))} \right) |\varepsilon \tilde{y}|^{-2} w^2 \sqrt{|g^\varepsilon|} dy}, \\ &= \frac{\int_{\mathbb{R}^N} (g^\varepsilon)^{ij} \partial_i w \partial_j w \sqrt{|g^\varepsilon|} dy}{\int_{\mathbb{R}^N} \left(q(Y(\varepsilon y)) + C_0 |\varepsilon \tilde{y}|^2 \sqrt{1-q(f(\varepsilon \bar{y}))} \right) |\tilde{y}|^{-2} w^2 \sqrt{|g^\varepsilon|} dy}, \end{aligned}$$

where g^ε is the metric with component $g_{ij}^\varepsilon(y) = g_{ij}(\varepsilon y)$ with $(g^\varepsilon)^{ij}(y)$ denotes the component of the inverse matrix of g^ε and $|g^\varepsilon|$ stands for the determinant of g^ε .

Observe that the scaled metric g^ε expands a $g^\varepsilon = Id + O(\varepsilon)$ on the support of w by (4.29). In addition since q is of class C^1 , decreasing ε_0 if necessary, there exists $c > 0$ such that

$$1 - q(f(\varepsilon \bar{y})) \leq c\varepsilon \quad \forall \bar{y} \in \text{Supp} w \cap \mathbb{R}^k, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

by (4.28). From this we deduce that

$$|\varepsilon \tilde{y}|^2 \sqrt{1-q(f(\varepsilon \tilde{y}))} \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0,$$

uniformly in $y \in \text{Supp} w$. We then have from the dominated convergence theorem and using (4.27) together with (4.28)

$$\frac{(N-k-2)^2}{4} \leq \frac{1}{1+C_0} \frac{\int_{\mathbb{R}^N} |\nabla w|^2 dy}{\int_{\mathbb{R}^N} |\tilde{y}|^{-2} w^2 dy} \quad \forall w \in C_c^\infty(\mathbb{R}^{N-k} \setminus \{0\} \times \mathbb{R}^k).$$

This is in contradiction with the well know fact that

$$\inf_{w \in C_c^\infty(\mathbb{R}^{N-k} \setminus \{0\} \times \mathbb{R}^k)} \frac{\int_{\mathbb{R}^N} |\nabla w|^2 dy}{\int_{\mathbb{R}^N} |\tilde{y}|^{-2} w^2 dy} = \inf_{w \in C_c^\infty(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla w|^2 dy}{\int_{\mathbb{R}^N} |\tilde{y}|^{-2} w^2 dy} = \frac{(N-k-2)^2}{4}$$

because $N-k > 2$, see for instance [[25], Section 2.1.6] and [[26], Lemma 1.1]. \square

4.2 Existence

Proposition 4.4 *Let $1 \leq p < \frac{N-k+2}{N-k-2}$. Then if β is small, there exists $u \in L^p(\Gamma_\beta)$ satisfying*

$$(4.30) \quad \begin{cases} -\Delta u - \frac{(N-k-2)^2}{4} \delta^{-2} q u \geq u^p & \text{in } \Gamma_\beta \setminus \Gamma, \\ u > 0 & \text{in } \Gamma_\beta. \end{cases}$$

Proof. Set

$$u = \omega_0 - \omega_{-1} = \omega_0(1 - X_{-1}(\delta)).$$

Then by Lemma 4.3 there exists $C > 0$ such that

$$L_q u \geq 2\delta^{-2} X_{-3}(\delta) \delta^{-\alpha} - C X_1(\delta) \delta^{-\frac{3}{2}} \delta^{-\alpha} \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

Hence, provided β is small, we have $u > 0$ and

$$-\Delta u - \frac{(N-k-2)^2}{4} \delta^{-2} q u \geq \delta^{-2} X_{-5}(\delta) u \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

We thus want

$$\delta^{-2} X_{-5}(\delta) u \geq u^p \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

Or equivalently

$$\delta^{-2} X_{-5}(\delta) \delta^{-\alpha} (1 - X_{-1}(\delta)) \geq \delta^{-p\alpha} (1 - X_{-1}(\delta))^p \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

That is

$$(4.31) \quad \delta^{-2+(p-1)\alpha} X_{-5}(\delta) (1 - X_{-1}(\delta))^{1-p} \geq 1 \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

We observe that for $1 \leq p < \frac{N-k+2}{N-k-2}$ we have for every $x \in \Gamma_\beta \setminus \Gamma$

$$-2 + (p-1)\alpha(x) \leq -2 - (p-1) \frac{N-k-2}{2} < 0.$$

This implies that if β is small enough, (4.31) holds so that u satisfies (4.30). The fact that $u \in L^p(\Gamma_\beta)$ is easy to check, we skip the details. \square

Acknowledgments

This work is supported by the Alexander-von-Humboldt Foundation. The author would like to thank Professor Roberta Musina and Professor Tobias Weth for their useful comments and suggestions. We gratefully thank the referee for carefully reading this manuscript and who's comments have helped to improve considerably the former versions and also for taking the author's attention to [8].

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